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Fourier Collocation Method for Solving Nonlinear Klein–Gordon Equation

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A Fourier collocation scheme is proposed for solving the periodic problem of nonlinear Klein–Gordon equation. It keeps the energy discretely and is shown to be stable or generalized stable. Its convergence is investigated. Numerical results are also presented. © 1993 Academic Press, Inc.

I. INTRODUCTION

As we know, the Klein–Gordon equation is an important mathematical model in quantum mechanics [1]. It is of the form

$$\begin{aligned} &\frac{\partial^2 U}{\partial t^2}(x,t) - \Delta U(x,t) + bU(x,t) + g(U(x,t)) \\ &\qquad = f(x,t), \qquad x \in R^n, \quad t \in (0,T], \\ &\frac{\partial U}{\partial t}(x,0) = U_1(x), \qquad x \in R^n, \\ &U(x,0) = U_0(x), \qquad x \in R^n, \end{aligned} \tag{1.1}$$

where $x=(x_1, x_2, ..., x_n)$, $\Delta = \sum_{j=1}^{n} (\partial^2/\partial x_j^2)$, b is a real number, function g is defined as $g(z)=|z|^\alpha z$. We suppose that

$$\begin{aligned} &\alpha \geqslant 0, && \text{for} \quad n \leqslant 2, \\ &0 < \alpha \leqslant \frac{2}{n-2}, && \text{for} \quad n \geqslant 3. \end{aligned}$$

In this paper, we only discuss the periodic problem of (1.1). We assume that U_0, U_1 and f have period 2π for x_j, $1 \leqslant j \leqslant n$.

There are many papers in literature on the existence, uniqueness of the smooth or weak solution of (1.1), e.g., [2]. Under some conditions, (1.1) possesses a unique solution, and, furthermore, the smoother U_0, U_1, and f are, the smoother the solution U. Many authors considered also its numerical solution, e.g., Strauss and Vazquez [3] by finite difference method, Kuo and Vazquez [4] by a finite element method. But these approximations are only of second order of convergence in space directions. Thus, in [5] the authors of the present paper proposed a Fourier spectral scheme to solve (1.1), which is of "infinite order" of convergence, if the exact solution is infinitely differentiable. Numerical results there show also that the spectral scheme is more accurate than the similar finite difference scheme. However, because of the nonlinear term $g(U)$, it is very difficult to implement the spectral method strictly, except if α is a small integer. Hence, we discuss in this paper the collocation method to solve (1.1). As will be shown below, this method is stable or generalized stable and is also of "infinite order" of convergence. Besides it simulates the energy conservation law discretely. Most of all, it can be easily implemented for all α.

An outline of this paper is as follows. In Section II we construct the collocation scheme. A discrete energy conservation law is shown to be satisfied by the approximate solution. In Section III we present some numerical results. Several lemmas are listed in Section IV. The stability and convergence are analysed in Section V and Section VI, separately.

II. SCHEME

As usual [6], let $\Omega = (0, 2\pi)^n$,

$$L^q(\Omega) = \{v/v \text{ is Lebesgue measurable on } \Omega \text{ and } \|v\|_{L^q} < \infty\},$$

where

$$\|v\|_{L^q} = \begin{cases} \left(\int_\Omega |v|^q \, dx\right)^{1/q}, & \text{if} \quad 1 \leqslant q < \infty, \\ \operatorname{ess\,sup}_{x \in \Omega} |v(x)|, & \text{if} \quad q = \infty. \end{cases}$$

For $q=2$, we denote by $\|\cdot\|$ and $(\cdot,\cdot)$ the norm and the inner product of $L^q(\Omega)$, respectively. Let $v=(v_1, v_2, ..., v_n)$,

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$|\nu| = \nu_1 + \nu_2 + \cdots + \nu_n$. $D^\nu = \partial^{|\nu|}/\partial x_1^{\nu_1} \cdots \partial x_n^{\nu_n}$. For any non-negative integer m, define $H^0(\Omega) = L^2(\Omega)$ and

$$H^m(\Omega) = \{v / D^\nu v \in L^2(\Omega), 0 \leqslant |\nu| \leqslant m\},$$

equipped with the semi-norm $|\cdot|_m$ and the norm $\|\cdot\|_m$ as

$$|v|_m = \Big(\sum_{|\nu| = m} \|D^\nu v\|^2 \Big)^{1/2}, \qquad \|v\|_m = (\|v\|^2_{m-1} + |v|^2_m)^{1/2}.$$

For non-negative real number s, we define $H^s(\Omega)$ by the interpolation of spaces. Its norm and semi-norm are still denoted by $\|\cdot\|_s$ and $|\cdot|_s$. Let $C_p^\infty(\Omega)$ be the set of infinitely differentiable functions which are 2π periodic for all x_j. $H_p^s(\Omega)$ is the closure of $C_p^\infty(\Omega)$ in $H^s(\Omega)$.

Let $j = (j_1, j_2, ..., j_n)$ and $|j|_\infty = \max_{1 \leqslant l \leqslant n} |j_l|$. We define for any positive integer N,

$$S_N = \Big\{ v = \sum_{|j|_\infty \leqslant N} v_j e^{ij \cdot x} / v_j = \bar{v}_{-j} \text{ for } |j|_\infty \leqslant N, \quad v_j = v_{-j} \text{ for } |j|_\infty = N \Big\}.$$

S_N is actually the set of real-valued trigonometric polynomials of degree $\leqslant N$. Let $h = 2\pi/(2N+1)$, and

$$\Omega_N = \{x_j = (j_1 h, j_2 h, ..., j_n h) / 0 \leqslant j_l \leqslant 2N, l = 1, 2, ..., n\}.$$

We denote by P_c the interpolation operator from $C_p(\bar{\Omega})$ onto S_N; i.e., for any $v \in C_p(\bar{\Omega})$, $P_c v \in S_N$ satisfies

$$P_c v(x_j) = v(x_j) \qquad \forall x_j \in \Omega_N.$$

Parallel to the L^q-norm and L^2-inner product, we define the discrete L^q-norm and the discrete L^2-inner product associated with the above collocation points as

$$\|v\|_{L^q, N} = \begin{cases} \Big(h^n \sum_{x_j \in \Omega_N} |v(x_j)|^q \Big)^{1/q}, & \text{if } 1 \leqslant q < \infty, \\ \sup_{x_j \in \Omega_N} |v(x_j)|, & \text{if } q = \infty. \end{cases}$$

and

$$(v, w)_N = h^n \sum_{x_j \in \Omega_N} u(x_j)\, v(x_j).$$

It is not difficult to verify that [7]

$$P_c v = v \qquad \forall v \in S_N, \tag{2.1}$$

$$(v, w)_N = (P_c v, P_c w)_N = (P_c v, P_c w) \qquad \forall v, w \in C_p(\bar{\Omega}). \tag{2.2}$$

Let τ be the step size in time t, and $R_\tau = \{t = k\tau / 1 \leqslant k \leqslant [T/\tau]\}$. We introduce the following notations for time discretization:

$$\hat{v}(x, t) = \frac{1}{2}(v(x, t+\tau) + v(x, t-\tau)),$$

$$v_t(x, t) = \frac{1}{\tau}(v(x, t+\tau) - v(x, t)),$$

$$v_{\bar{t}}(x, t) = v_t(x, t-\tau),$$

$$v_{\hat{t}}(x, t) = \frac{1}{2}(v_t(x, t) + v_{\bar{t}}(x, t)),$$

$$v_{t\bar{t}}(x, t) = \frac{1}{\tau}(v_t(x, t) - v_t(x, t-\tau)).$$

It is easy to see that [8]

$$2(v_{\hat{t}}(t), \hat{v}(t)) = (\|v(t)\|^2)_{\hat{t}}, \tag{2.3}$$

$$2(v_{\hat{t}}(t), v_{t\bar{t}}(t)) = (\|v_{\bar{t}}(t)\|^2)_t. \tag{2.4}$$

Now we construct the collocation scheme. It is well known that the solution of (1.1) possesses the energy conservation

$$E(U, t) = E(U, 0) + 2\int_0^t \Big(\frac{\partial U}{\partial t'}(t'), f(t') \Big)\, dt', \tag{2.5}$$

where $p = \alpha + 2$,

$$E(U, t) = \Big\| \frac{\partial U}{\partial t}(t) \Big\|^2 + |U(t)|_1^2 + b\, \|U(t)\|^2 + \frac{2}{p} \|U(t)\|^p_{L^p}.$$

A key point to discretize (1.1) is to simulate the above conservation reasonably. To this end, we define

$$G(v(x, t)) = \int_0^1 g(\sigma v(x, t+\tau) + (1-\sigma)\, v(x, t-\tau))\, d\sigma.$$

Obviously, $G(v(x, t))$ is an approximation of $g(v(x, t))$. Furthermore, since

$$g(z) = \frac{1}{p} \frac{d}{dz} |z|^p,$$

we have

$$2 v_{\hat{t}}(t)\, G(v(t)) = \frac{1}{\tau p}(|v(x, t+\tau)|^p - |v(x, t-\tau)|^p),$$

and thus

$$2(G(v(t)), v_{\hat{t}}(t)) = \frac{1}{p}(\|v(t)\|^p_{L^p})_{\hat{t}}. \tag{2.6}$$

The Fourier collocation scheme for solving (1.1) is to find $u(x, t) \in S_N$ for all $t \in R_\tau$ such that

$$\begin{aligned} &u_{t\bar{t}}(x_j, t) - \Delta\hat{u}(x_j, t) + b\hat{u}(x_j, t) + G(u(x_j, t)) \\ &\quad = \hat{f}(x_j, t), \qquad x_j \in \Omega_N,\ t \in R_\tau, \\ &u_t(x_j, 0) = u_1(x_j), \qquad x_j \in \Omega_N, \\ &u(x_j, 0) = U_0(x_j), \qquad x_j \in \Omega_N, \end{aligned} \tag{2.7}$$

where

$$u_1(x_j) = U_1(x_j) + \frac{\tau}{2}(\Delta U_0(x_j) - bU_0(x_j) - g(U_0(x_j)) + f(x_j, 0)).$$

It is not difficult to show that the solution $u(x, t)$ of (2.6) satisfies a discrete energy conservation law similar to (2.5). In fact, by multiplying the first equation of (2.7) by $2u_{\hat{t}}(x_j, t)$ and summing up for all $x_j \in \Omega_N$, we obtain from (2.1), (2.3), (2.4), and (2.6) that

$$(\|u_t(t)\|^2)_{\bar{t}} + (|u(t)|_1^2)_{\hat{t}} + b(\|u(t)\|^2)_{\hat{t}} + \frac{2}{p}(\|u(t)\|_{L^p,N}^p)_{\hat{t}} = 2(u_{\hat{t}}(t), \hat{f}(t))_N.$$

A summation of the above equality for all $t' \in R_\tau$, $t' \leqslant t-\tau$, yields that

$$E^*(u, t) = E^*(u, 0) + 2\tau \sum_{t' \leqslant t-\tau} (u_{\hat{t}}(t'), \hat{f}(t'))_N, \tag{2.8}$$

where

$$\begin{aligned} E^*(u, t) = \|u_t(t)\|^2 &+ \frac{1}{2}(|u(t)|_1^2 + |u(t-\tau)|_1^2) \\ &+ \frac{b}{2}(\|u(t)\|^2 + \|u(t-\tau)\|^2) \\ &+ \frac{1}{p}(\|u(t)\|_{L^p,N}^p + \|u(t-\tau)\|_{L^p,N}^p). \end{aligned}$$

Clearly (2.8) is a reasonable analogue of (2.5).

III. NUMERICAL RESULTS

In this section, we present some numerical results. For comparison, we solve (1.1) also by a finite difference scheme [8]. Define

$$e_j = (\underbrace{0, ..., 0, 1}_{j}, 0, ..., 0),$$

and

$$\Delta_h v(x, t) = \sum_{j=1}^{n} \frac{1}{h^2}(v(x+he_j, t) - 2v(x, t) + v(x-he_j, t)).$$

The finite difference scheme is

$$\begin{aligned} &u_{t\bar{t}}(x_j, t) - \Delta_h\hat{u}(x_j, t) + b\hat{u}(x_j, t) + G(u(x_j, t)) \\ &\quad = \hat{f}(x_j, t), \qquad x_j \in \Omega_N, \quad t \in R_\tau, \\ &u_t(x_j, 0) = u_1(x_j), \qquad x_j \in \Omega_N, \\ &u(x_j, 0) = U_0(x_j), \qquad x_j \in \Omega_N, \end{aligned} \tag{3.1}$$

where

$$u_1(x_j) = U_1(x_j) + \frac{\tau}{2}(\Delta_h U_0(x_j) - bU_0(x_j) - g(U_0(x_j)) + f(x_j, 0)).$$

For describing the error, we define

$$\tilde{E}_h(t) = \frac{\|U(t) - u(t)\|_{L^2,N}}{\|U(t)\|_{L^2,N}}.$$

For simplicity, we take $n = 1$ and the test function

$$U(x, t) = e^{A\sin(x+Dt)},$$

where A and D are parameters. In all the calculations, we fix $T = 1$ and $b = 1$.

We consider two pairs of the parameter: (i) $\alpha = 2.5$, $A = 2.0$, $D = 1.0$, which corresponds to strong nonlinearity and slow wave movement; (ii) $\alpha = 0.5$, $A = 1.0$, $D = 6.2832$, which corresponds to weak nonlinearity and rapid wave movement. We find that

(1) In both cases, the Fourier collocation method (scheme (2.7)) gives more accurate results than the finite difference method (scheme (3.1)).

TABLE I

$\tilde{E}_h(1.0)$

	Scheme (2.7)		Scheme (3.1)	
N	$\tau = 10^{-2}$	$\tau = 10^{-3}$	$\tau = 10^{-2}$	$\tau = 10^{-3}$
4	0.26236E − 2	0.26351E − 2	0.12173E − 1	0.12303E − 1
8	0.14767E − 3	0.17352E − 5	0.34388E − 2	0.34785E − 2
16	0.14763E − 3	0.15123E − 5	0.85270E − 3	0.87579E − 3
32	—	—	0.23912E − 3	0.22425E − 3
64	—	—	0.14501E − 3	0.56752E − 4

Note. $\alpha = 2.5$; $A = 2.0$; $D = 1.0$.

TABLE II

$\tilde{E}_h(1.0)$

	Scheme (2.7)		Scheme (3.1)	
N	$\tau=10^{-2}$	$\tau=10^{-3}$	$\tau=10^{-2}$	$\tau=10^{-3}$
4	0.61151E−2	0.63403E−4	0.95757E−2	0.30394E−2
8	0.61139E−2	0.60526E−4	0.70460E−2	0.88223E−3
16	0.61139E−2	0.60528E−4	0.63592E−2	0.27795E−3
32	—	—	0.61770E−2	0.11643E−3
64	—	—	0.61300E−2	0.74694E−4

Note. $\alpha=0.5$; $A=1.0$; $D=6.2832$.

(2) Spectral accuracy in x direction is observed in scheme (2.7), if t direction is well resolved, see Table I for $\tau=10^{-3}$, $N=8$ and 16. However, since scheme (2.7) is of a much higher convergence rate in x direction ($O(N^{-S})$) than that in the t direction ($O(\tau^2)$), in most cases it is the insufficient resolution in the t direction that hinders the overall accuracy of the scheme, especially when the wave moves rapidly (see Tables I and II). Hence a high resolution technique in the t direction seems imperative to spectral method.

(3) Second-order convergence in the t direction for scheme (2.7) is obvious, except only for the case that N is too small to resolve the solution ($N=4$ for parameter setting (i)).

(4) For the finite difference scheme (3.1), the second order convergence in both x and t directions is clearly shown, see Tables I and II.

IV. LEMMAS

In order to analyse the stability and convergence, we introduce in this section some notations and lemmas. Let B be a Banach space. Define

$$C(0, T; B)=\{v/v\colon [0, T]\to B \text{ is strongly continuous}\},$$

equipped with the norm

$$\|v\|_{C(0,T;B)}=\max_{0\leqslant t\leqslant T}\|v(t)\|_B.$$

Furthermore,

$$C^m(0, T; B)=\left\{v\Big/\frac{\partial^k v}{\partial t^k}\in C(0, T; B), 0\leqslant k\leqslant m\right\},$$

and

$$\|v\|_{C^m(0,T;B)}=\max_{0\leqslant k\leqslant m}\left\|\frac{\partial^k v}{\partial t^k}\right\|_{C(0,T;B)}.$$

LEMMA 1 [7]. *Assume $s>n/2$, and $0\leqslant\mu\leqslant s$. There exists a positive constant c independent of N, such that for all $v\in H_p^s(\Omega)$,*

$$\|v-P_cv\|_\mu\leqslant cN^{\mu-s}|v|_s.$$

LEMMA 2 [9]. *For all $v\in S_N$,*

$$|v|_1\leqslant n^{1/2}N\|v\|.$$

LEMMA 3. *There exists a positive constant c independent of τ, such that for all $v\in C^4(0, T; L^\infty(\Omega))$,*

$$\|\hat{v}(t)-v(t)\|_{L^2,N}\leqslant c\tau^2\|v\|_{C^2(0,T;L^\infty)},$$

$$\left\|v_{\hat{t}t}(t)-\frac{\partial^2 v}{\partial t^2}(t)\right\|_{L^2,N}\leqslant c\tau^2\|v\|_{C^4(0,T;L^\infty)},$$

$$\left\|v_{\hat{t}}(t)-\frac{\partial v}{\partial t}(t)-\frac{\tau}{2}\frac{\partial^2 v}{\partial t^2}(t)\right\|_{L^2,N}\leqslant c\tau^2\|v\|_{C^3(0,T;L^\infty)}.$$

LEMMA 4. *There exists a positive constant c independent of τ, such that for all $v\in C^1(0, T; L^\infty)$,*

$$\|P_c[G(P_cv(t))-\hat{g}(v(t))]\|\leqslant\begin{cases}c\tau\|v\|^{\alpha+1}_{C^1(0,T;L^\infty)} & \text{for } 0\leqslant\alpha<1,\\ c\tau^2\|v\|^{\alpha+1}_{C^1(0,T;L^\infty)} & \text{for } \alpha\geqslant 1.\end{cases}$$

Proof. Consider $G(P_cv(x,t))$ at each collocation points $x\in\Omega_N$, where it coincides with $G(v(x,t))$. By Taylor's expansion

$$\begin{aligned}g(\sigma v(x, t+\tau)+(1-\sigma)\,v(x, t-\tau))&=g(v(x, t-\tau))+\sigma[v(x, t+\tau)-v(x, t-\tau)]\\&\quad\times\frac{dg}{dz}[\theta(\sigma)\,v(x, t+\tau)+(1-\theta(\sigma))\,v(x, t-\tau)],\end{aligned}$$

where $0\leqslant\theta(\sigma)\leqslant\sigma$. Thus the first mean value theorem leads to

$$G[v(x,t)]=g(v(x, t-\tau))+\frac{1}{2}[v(x, t+\tau)-v(x, t-\tau)]\times\frac{dg}{dz}[\theta_1 v(x, t+\tau)+(1-\theta_1)\,v(x, t-\tau)],\quad 0\leqslant\theta_1\leqslant 1.$$

Similarly,

$$G[v(x,t)]=g(v(x, t+\tau))-\frac{1}{2}[v(x, t+\tau)-v(x, t-\tau)]\times\frac{dg}{dz}[\theta_2 v(x, t+\tau)+(1-\theta_2)\,v(x, t-\tau)],\quad 0\leqslant\theta_2\leqslant 1.$$

Note that

$$\frac{dg}{dz}(z)=(\alpha+1)\,|z|^{\alpha}$$

and

$$|v(x,t+\tau)-v(x,t-\tau)|$$
$$\leqslant 2\tau\left|\frac{\partial v}{\partial t}(x,t_0)\right|,\qquad t-\tau\leqslant t_0\leqslant t+\tau. \qquad (4.1)$$

Therefore,

$$\|P_c[G(P_c v(t))-\hat{g}(v(t))]\|\leqslant c\tau\,\|v\|^{\alpha}_{C(0,T;L^\infty)}\left\|\frac{\partial v}{\partial t}\right\|_{C(0,T;L^\infty)}.$$

If $\alpha\geqslant 1$, then we have from the expression of remainder of trapezoidal integration,

$$|G(v(x,t))-\hat{g}(v(x,t))|$$
$$=\frac{1}{12}\left|\frac{d^2g}{dz^2}(\theta_0 v(x,t+\tau)+(1-\theta_0)\,v(x,t-\tau)\right|$$
$$\times|v(x,t+\tau)-v(x,t-\tau)|^2,$$

where $0\leqslant\theta_0\leqslant 1$. Moreover,

$$\frac{d^2g}{dz^2}(z)=(\alpha+1)\,\alpha\,|z|^{\alpha-2}\,z,$$

which, together with (4.1), yields the desired conclusion.

LEMMA 5. *Assume* $1\leqslant q\leqslant\infty$, *and* $H^s(\Omega)\subsetneq L^q(\Omega)$. *Then there exists a positive constant c independent of N, such that for all* $v\in S_N$,

$$\|v\|_{L^q,N}\leqslant c\,\|v\|_s.$$

Proof. Suppose

$$v(x)=\sum_{|j|_\infty\leqslant N} v_j e^{ij\cdot x}.$$

Define

$$Rv(x)=\sum_{|j|_\infty\leqslant N} r_j v_j e^{ij\cdot x},$$

where

$$r_j=\prod_{l=1}^{n}\frac{j_l h/2}{\sin(j_l h/2)}.$$

First we prove that

$$\|v\|_{L^q,N}\leqslant\|Rv\|_{L^q}. \qquad (4.2)$$

It is not difficult to see that

$$v(x_v)=\sum_{|j|_\infty\leqslant N} v_j e^{ij\cdot x_v}$$
$$=h^{-n}\int_{|x-x_v|_\infty\leqslant h/2}\left(\sum_{|j|_\infty\leqslant N} r_j v_j e^{ij\cdot x}\right)dx$$
$$=h^{-n}\int_{|x-x_v|_\infty\leqslant h/2} Rv(x)\,dx.$$

Hence we have from Holder's inequality that

$$|v(x_v)|\leqslant h^{-n}\left[\int_{|x-x_v|_\infty\leqslant h/2}|Rv(x)|^q\,dx\right]^{1/q}$$
$$\times\left[\int_{|x-x_v|_\infty\leqslant h/2}1\,dx\right]^{1/q'},\qquad \frac{1}{q}+\frac{1}{q'}=1$$
$$=h^{-n/q}\left[\int_{|x-x_v|_\infty\leqslant h/2}|Rv(x)|^q\,dx\right]^{1/q}.$$

Thus

$$\|v\|^q_{L^q,N}\leqslant h^n\sum_{x_v\in\Omega_N}|v(x_v)|^q$$
$$\leqslant\sum_{x_v\in\Omega_N}\int_{|x-x_v|_\infty\leqslant h/2}|Rv(x)|^q\,dx$$
$$=\int_\Omega|Rv(x)|^q\,dx=\|Rv\|^q_{L^q}.$$

Next we prove that

$$\|Rv\|_{L^q}\leqslant c\,\|v\|_s. \qquad (4.3)$$

Clearly by $H^s(\Omega)\subsetneq L^q(\Omega)$, we have

$$\|Rv\|_{L^q}\leqslant c\,\|Rv\|_s.$$

On the other hand, since $|j|_\infty\leqslant N$, $-\pi/2\leqslant j_l h/2\leqslant\pi/2$ $(1\leqslant l\leqslant n)$, thus

$$|r_j|=\prod_{l=1}^{n}\left|\frac{j_l h/2}{\sin(j_l h/2)}\right|\leqslant(2\pi)^n.$$

Consequently by the equivalence of the norms in $H^s_p(\Omega)$,

$$\|Rv\|_s\leqslant c\left[\sum_{|j|_\infty\leqslant N}(1+|j|^2)^s\,|r_j v_j|^2\right]^{1/2}$$
$$\leqslant c\left[\sum_{|j|_\infty\leqslant N}(1+|j|^2)^s\,|v_j|^2\right]^{1/2}$$
$$\leqslant c\,\|v\|_s.$$

The combination of (4.2) and (4.3) leads to the conclusion of this lemma.

LEMMA 6. *There exists a positive constant c independent of N and τ, such that for all v, w ∈ C(0, T; S_N),*

$$G(v(x,t)+w(x,t)) = G(v(x,t)) + R(x,t),$$

with

$$\|R(t)\|^2_{L^2,N} \leqslant c(\|v\|^{2\alpha}_{C(0,T;H^1)} + \|w\|^{2\alpha}_{C(0,T;H^1)}) \times (\|w(t+\tau)\|^2_1 + \|w(t-\tau)\|^2_1).$$

Proof. Let

$$V(\sigma) = \sigma v(x, t+\tau) + (1-\sigma)\, v(x, t-\tau),$$
$$W(\sigma) = \sigma W(x, t+\tau) + (1-\sigma)\, W(x, t-\tau).$$

Then by Taylor's expansion and that [10]

$$(a_1+a_2)^\alpha \leqslant c(a_1^\alpha + a_2^\alpha) \qquad \forall a_1, a_2 \geqslant 0,$$

we have

$$\begin{aligned} |R(x,t)| &\leqslant \int_0^1 |g(V(\sigma)+W(\sigma)) - g(V(\sigma))|\, d\sigma \\ &= (\alpha+1)\int_0^1 |V(\sigma)+\theta(\sigma)\, W(\sigma)|^\alpha\, |W(\sigma)|\, d\sigma, \\ &\quad 0 \leqslant \theta(\sigma) \leqslant 1 \\ &\leqslant (|v(x,t+\tau)|^\alpha + |v(x,t-\tau)|^\alpha + |w(x,t+\tau)|^\alpha \\ &\quad + |w(x,t-\tau)|^\alpha)(|w(x,t+\tau)| + |w(x,t-\tau)|). \end{aligned}$$

Taking $\beta = \max(3/2, n/2)$, we have from Holder's inequality that

$$\begin{aligned} \|R(t)\|_{L^2,N} &\leqslant c(\|v(t+\tau)\|^{2\alpha}_{L^{2\alpha\beta},N} + \|v(t-\tau)\|^{2\alpha}_{L^{2\alpha\beta},N} \\ &\quad + \|w(t+\tau)\|^{2\alpha}_{L^{2\alpha\beta},N} + \|w(t-\tau)\|^{2\alpha}_{L^{2\alpha\beta},N}) \\ &\quad \times (\|w(t+\tau)\|^2_{L^{2\beta/(\beta-1)},N} + \|w(t-\tau)\|^2_{L^{2\beta/(\beta-1)},N}). \end{aligned} \tag{4.4}$$

Since $H^1(\Omega) \subsetneq L^{2\alpha\beta}(\Omega)$ and $H^1(\Omega) \subsetneq L^{2\beta/(\beta-1)}(\Omega)$, we can complete the proof of this lemma by Lemma 5.

Next we consider a special case,

$$\begin{aligned} &1 \leqslant \alpha \leqslant 2 &&\text{for } n=1, \\ &1 \leqslant \alpha < 2 &&\text{for } n=2, \\ &1 \leqslant \alpha \leqslant \frac{4}{n} &&\text{for } n \geqslant 3. \end{aligned} \tag{4.5}$$

In this case, we can improve the result of the previous lemma.

LEMMA 7. *Assume α satisfies* (4.5). *Then for all v, w ∈ C(0, T; S_N), we have*

$$G(v(x,t)+w(x,t)) = G(v(x,t)) + G(w(x,t)) + R(x,t)$$

with

$$\|R(t)\|^2_{L^2,N} \leqslant d(v)(\|w(t+\tau)\|^2_1 + \|w(t-\tau)\|^2_1 + \|w(t+\tau)\|^p_{L^p,N} + \|w(t-\tau)\|^p_{L^p,N}),$$

where d(v) is a positive constant depending only on α, $\|v(t+\tau)\|_1$, and $\|v(t-\tau)\|_1$.

Proof. Define $V(\sigma)$ and $W(\sigma)$ as in the proof of last lemma. Then by Taylor's expansion

$$\begin{aligned} |V(\sigma)+W(\sigma)|^\alpha &= |V(\sigma)|^\alpha + \alpha\, |V(\sigma)+\theta_1 W(\sigma)|^{\alpha-2} \\ &\quad \times (V(\sigma)+\theta_1 W(\sigma))\, W(\sigma) \\ |V(\sigma)+W(\sigma)|^\alpha &= |W(\sigma)|^\alpha + \alpha\, |W(\sigma)+\theta_2 V(\sigma)|^{\alpha-2} \\ &\quad \times (W(\sigma)+\theta_2 V(\sigma))\, V(\sigma), \end{aligned}$$

where $0 \leqslant \theta_1, \theta_2 \leqslant 1$. Hence

$$g(V(\sigma)+W(\sigma)) = g(V(\sigma)) + g(W(\sigma)) + R(\sigma),$$

where

$$\begin{aligned} |R(\sigma)| &\leqslant c(|v(x,t+\tau)|^\alpha + |v(x,t-\tau)|^\alpha) \\ &\quad \times (|w(x,t+\tau)| + |w(x,t-\tau)|) \\ &\quad + c(|w(x,t+\tau)|^\alpha + |w(x,t-\tau)|^\alpha) \\ &\quad \times (|v(x,t+\tau)| + |v(x,t-\tau)|). \end{aligned}$$

We bound the terms on the right hand side in the above inequality separately. Clearly, by taking $\beta = \max(3/2, n/2)$, we have from Holder's inequality and Lemma 5 that

$$\begin{aligned} &\|\,|v(t+\tau)|^\alpha\, w(t+\tau)\|^2_{L^2,N} \\ &\quad \leqslant \|v(t+\tau)\|^{2\alpha}_{L^{2\alpha\beta},N}\, \|w(t+\tau)\|^2_{L^{2\beta/(\beta-1)},N} \\ &\quad \leqslant c\, \|v(t+\tau)\|^{2\alpha}_1\, \|w(t+\tau)\|^2_1. \end{aligned}$$

Similar treatment can be applied to $|v(x,t-\tau)|^\alpha\, |w(x,t+\tau)|$, etc. Next we consider $|w(x,t-\tau)|^\alpha\, |v(x,t+\tau)|$ and its similar terms. If $n=1$ and 2, then $H^1(\Omega) \subsetneq L^{2(\alpha+2)/(2-\alpha)}(\Omega)$. Thus Holder's inequality and Lemma 5 lead to

$$\begin{aligned} &\|\,|w(t+\tau)|^\alpha\, v(t+\tau)\|^2_{L^2,N} \\ &\quad \leqslant \|v(t+\tau)\|^2_{L^{2(\alpha+2)/(2-\alpha)},N}\, \|w(t+\tau)\|^{2\alpha}_{L^{\alpha+2},N} \\ &\quad \leqslant c\, \|v(t+\tau)\|^2_1\, \|w(t+\tau)\|^{2\alpha}_{L^p,N}. \end{aligned}$$

Note that [10] for $q, q' \geqslant 1$ satisfying $1/q+1/q'=1$,

$$a_1 a_2 \leqslant \frac{a_1^q}{q}+\frac{a_2^{q'}}{q'} \qquad \forall a_1, a_2 \geqslant 0. \tag{4.6}$$

Hence we obtain from Lemma 5 that

$$\begin{aligned}\|w(t+\tau)\|_{L^p,N}^{2\alpha} &\leqslant \frac{2-\alpha}{\alpha}\|w(t+\tau)\|_{L^p,N}^2 \\ &\quad+\frac{2(\alpha-1)}{\alpha}\|w(t+\tau)\|_{L^p,N}^p \\ &\leqslant c(\|w(t+\tau)\|_1^2+\|w(t+\tau)\|_{L^p,N}^p),\end{aligned}$$

which gives the conclusion for $n=1, 2$ and $1 \leqslant \alpha < 2$. If $n=1$ and $\alpha=2$, we can easily check that the above inequality holds too.

If $n=3$ and $1 \leqslant \alpha \leqslant 4/n$, then we take $\beta = n\alpha/(n\alpha+2\alpha-4)$. So Holder's inequality leads to

$$\begin{aligned}&\| |w(t+\tau)|^\alpha v(t+\tau)\|_{L^2,N}^2 \\ &\quad\leqslant \|w(t+\tau)\|_{L^{2\alpha\beta},N}^{2\alpha}\|v(t+\tau)\|_{L^{2\beta/(\beta-1)},N}^2.\end{aligned}$$

It is easy to see that $H^1(\Omega) \subsetneq L^{2\beta/(\beta-1)}(\Omega)$, since $\alpha \leqslant 4/n$. Furthermore, we have also from Holder's inequality that

$$\begin{aligned}&\|w(t+\tau)\|_{L^{2\alpha\beta},N}^{2\alpha} \\ &\quad\leqslant \|w(t+\tau)\|_{L^{\alpha+2},N}^{2(\alpha+2)(\alpha-1)/\alpha}\|w(t+\tau)\|_{L^{2n/(n-2)},N}^{2(2-\alpha)/\alpha} \\ &\quad\leqslant c\,\|w(t+\tau)\|_{L^p,N}^{2(\alpha+2)(\alpha-1)/\alpha}\|w(t+\tau)\|_1^{2(2-\alpha)/\alpha},\end{aligned}$$

where we have used $H^1(\Omega) \subsetneq L^{2n/(n-2)}(\Omega)$ and Lemma 5. Still by (4.6), we obtain that

$$\|w(t+\tau)\|_{L^{2\alpha\beta},N}^{2\alpha} \leqslant c(\|w(t+\tau)\|_{L^p,N}^p+\|w(t+\tau)\|_1^2).$$

Similarly we can bound $\| |w(t+\tau)|^\alpha v(t-\tau)\|$, etc. Thus we complete the proof.

V. STABILITY

First we derive an a priori estimation for the approximate solution $u(t)$ of (2.7). Assume $\tau N \leqslant r < \infty$. Hereafter we denote by c a general positive constant independent of τ, N, and u. By the discrete energy conservation (2.8), we need only to bound the initial values $E^*(u,\tau)$, $(b/2)(\|u(t)\|^2+\|u(t-\tau)\|^2)$ and $2\tau\sum_{t' \leqslant t-\tau}(u_{\bar t}(t'), \hat f(t'))_N$. We have from Lemma 3 that

$$\|u(\tau)\|_1 \leqslant \|u(0)\|_1 + c\tau N\,\|u_t(0)\|_1 \leqslant \|u(0)\|_1 + c\,\|u_t(0)\|. \tag{5.1}$$

Since α satisfies (2.1), $H^1(\Omega) \subsetneq L^p(\Omega)$. Thus we have from Lemma 5 that

$$\begin{aligned}\|u(0)\|_{L^p,N}+\|u(\tau)\|_{L^p,N} &\leqslant c(\|u(0)\|_1+\|u(\tau)\|_1) \\ &\leqslant c(\|u(0)\|_1+\|u_t(0)\|).\end{aligned}$$

On the other hand,

$$\begin{aligned}u^2(x,t) &= \Big(u_0(x)+\tau\sum_{t' \leqslant t} u_{\bar t}(x,t')\Big)^2 \\ &\leqslant 2u_0^2(x)+2t\tau\sum_{t' \leqslant t} u_{\bar t}^2(x,t'),\end{aligned}$$

which implies

$$\|u(t)\|^2 \leqslant 2\,\|u_0\|^2+2t\tau\sum_{t' \leqslant t}\|u_{\bar t}(t')\|^2.$$

Hence

$$\begin{aligned}&\left|\frac{b}{2}(\|u(t)\|^2+\|u(t-\tau)\|^2)\right| \\ &\quad\leqslant |b|\,t\tau\,\|u_{\bar t}(t)\|^2+2\,|b| \\ &\quad\quad\times\Big(\|u(0)\|^2+t\tau\sum_{t' \leqslant t-\tau}\|u_{\bar t}(t')\|^2\Big).\end{aligned} \tag{5.2}$$

It is not difficult to show that

$$\begin{aligned}&\Big|2\tau\sum_{t' \leqslant t-\tau}(\hat f(t'), u_{\bar t}(t'))_N\Big| \\ &\quad\leqslant \tau\,\|u_{\bar t}(t)\|^2+2\tau\sum_{t' \leqslant t-\tau}(\|u_{\bar t}(t')\|^2+\|\hat f(t')\|_{L^2,N}^2).\end{aligned} \tag{5.3}$$

Consequently, we derive from (2.8) that

$$\begin{aligned}&(1-\tau-|b|\,t\tau)\,\|u_{\bar t}(t)\|^2+\frac{1}{2}(\|u(t)\|_1^2+\|u(t-\tau)\|_1^2) \\ &\quad+\frac{1}{p}(\|u(t)\|_{L^p,N}^p+\|u(t-\tau)\|_{L^p,N}^p) \\ &\quad\leqslant \rho(u_0,u_1,f)+(1+2\,|b|\,t)\tau\sum_{t' \leqslant t-\tau}\|u_{\bar t}(t')\|^2,\end{aligned} \tag{5.4}$$

where

$$\begin{aligned}\rho(u_0,u_1,f) &= c(\|u_0\|_1^2+\|u_0\|_1^p+\|u_1\|^2+\|u_1\|^p) \\ &\quad+c\tau\sum_{t' \leqslant T}\|f(t')\|_{L^2,N}^2.\end{aligned}$$

Suppose τ is sufficiently small and define

$$E^N(u, t) = \|u_{\bar{t}}(t)\|^2 + \|u(t)\|_1^2 + \|u(t)\|_{L^p,N}^p.$$

Then by applying Gronwall's inequality [8] to (5.4), we obtain that

$$E^N(u, t) \leqslant c\rho(u_0, u_1, f)\, e^{ct}. \tag{5.5}$$

Remark 5.1. If $U_0, U_1 \in H_p^s(\Omega), f \in C(0, T; H_p^s(\Omega))$ with $s > \max(1, n/2)$, then we conclude by Lemma 1 that $\rho(u_0, u_1, f) \leqslant M$, M being a positive constant depending only on $\|U_0\|_s$, $\|U_1\|_s$, and $\|f\|_{C(0,T;H^s)}$.

Now we begin to study the stability of scheme (2.7). Suppose that u_0, u_1, and $P_c f$ in (2.7) have errors $\tilde{u}_0$, $\tilde{u}_1$, and $\tilde{f}$, respectively, which induce the error $\tilde{u}(t)$ in $u(t)$. Then they satisfy the equation

$$\begin{aligned} &\tilde{u}_{t\bar{t}}(x_j, t) - \Delta\hat{\tilde{u}}(x_j, t) + b\hat{\tilde{u}}(x_j, t) + P_c\tilde{G}(x_j, t) \\ &\quad = \hat{\tilde{f}}(x_j, t), \qquad x_j \in \Omega_N, \quad t \in R_\tau, \\ &\tilde{u}_t(x_j, 0) = \tilde{u}_1(x_j), \qquad x_j \in \Omega_N, \\ &\tilde{u}(x_j, 0) = \tilde{u}_0(x_j), \qquad x_j \in \Omega_N, \end{aligned} \tag{5.6}$$

where

$$\tilde{G}(x, t) = G(u(x, t) + \tilde{u}(x, t)) - G(u(x, t)).$$

By taking the discrete inner product with $2\tilde{u}_{\hat{t}}(x, t)$ in the first equation of (5.6), we have from (2.2), (2.3), and (2.4) that

$$\begin{aligned} &(\|\tilde{u}_{\bar{t}}(t)\|^2)_t + (|\tilde{u}(t)|_1^2 + b\,\|\tilde{u}(t)\|^2)_{\hat{t}} \\ &\quad + 2(P_c\tilde{G}(t), \tilde{u}_{\hat{t}}(t)) = 2(\hat{\tilde{f}}(t), \tilde{u}_{\hat{t}}(t))_N. \end{aligned} \tag{5.7}$$

Let $d(\tilde{u})$ and $d(u)$ be two positive constants depending only on $\|\tilde{u}\|_{C(0,T;H^1)}$ and $\|u\|_{C(0,T;H^1)}$, respectively. Then we obtain from Lemma 6 that

$$\begin{aligned} |2(P_c\tilde{G}(t), \tilde{u}_{\hat{t}}(t))| &\leqslant \|P_c\tilde{G}(t)\|\,(\|\tilde{u}_{\bar{t}}(t+\tau)\|^2 + \|\tilde{u}_{\bar{t}}(t)\|^2) \\ &\leqslant \tfrac{1}{2}(\|\tilde{u}_{\bar{t}}(t+\tau)\|^2 + \|\tilde{u}_{\bar{t}}(t)\|^2) \\ &\quad + (d(u) + d(\tilde{u})) \\ &\quad \times (\|\tilde{u}(t+\tau)\|^2 + \|\tilde{u}(t+\tau)\|^2). \end{aligned} \tag{5.8}$$

By an argument similar to the derivation of (5.4), together with (5.8), we obtain from (5.7) that

$$\begin{aligned} &(1 - c\tau)\,\|\tilde{u}_{\bar{t}}(t)\|^2 + (\tfrac{1}{2} - \tau d(u) - \tau d(\tilde{u})) \\ &\quad \times (\|\tilde{u}(t)\|_1^2 + \|\tilde{u}(t-\tau)\|_1^2) \\ &\leqslant \tilde{\rho}_1(\tilde{u}_0, \tilde{u}_1, \tilde{f}) + (c + d(u) + d(\tilde{u}))\tau \\ &\quad \times \sum_{t' \leqslant t-\tau} (\|\tilde{u}_{\bar{t}}(t')\|^2 + \|\tilde{u}(t')\|_1^2), \end{aligned} \tag{5.9}$$

where

$$\begin{aligned} \tilde{\rho}_1(\tilde{u}_0, \tilde{u}_1, \tilde{f}) &= (c + \tau d(u_0) + \tau d(\tilde{u}_0))(\|\tilde{u}_0\|_1^2 \\ &\quad + \|\tilde{u}_t(0)\|^2) + c\tau \sum_{t' \leqslant T} \|\tilde{f}(t')\|_{L^2,N}^2. \end{aligned}$$

On the other hand, by the a priori estimation we have

$$\|u\|_{C(0,T;H^1)} \leqslant c\rho_0(u_0, u_1, f)\, e^{cT}$$

and

$$\|u + \tilde{u}\|_{C(0,T;H^1)} \leqslant c\rho_0(u_0 + \tilde{u}_0, u_1 + \tilde{u}_1, f + \tilde{f})\, e^{cT}.$$

Thus if $\tilde{\rho}_1(\tilde{u}_0, \tilde{u}_1, f) \leqslant M_0$ for certain $M_0 > 0$, then we conclude that $\rho_0(u_0 + \tilde{u}_0, u_1 + \tilde{u}_1, f + \tilde{f})$, and furthermore $d(\tilde{u})$, are bounded above by a positive constant depending only on $\rho_0(u_0, u_1, f)$ and M_0. Consequently if τ is sufficiently small, then (5.9) implies that

$$\begin{aligned} \|\tilde{u}_{\bar{t}}(t)\|^2 + \|\tilde{u}(t)\|_1^2 &\leqslant M_1\tilde{\rho}_1(\tilde{u}_0, \tilde{u}_1, \tilde{f}) \\ &\quad + M_2\tau \sum_{t' \leqslant t-\tau} (\|\tilde{u}_{\bar{t}}(t')\|^2 + \|\tilde{u}(t')\|_1^2). \end{aligned}$$

Finally by Gronwall's inequality we obtain the following theorem.

THEOREM 5.1. *Assume that $\tau N \leqslant r < \infty$, and N is sufficiently large. $\tilde{\rho}_1(\tilde{u}_0, \tilde{u}_1, \tilde{f}) \leqslant M_0$. Then we have for all $t \in R_\tau$ that*

$$\|\tilde{u}_{\bar{t}}(t)\|^2 + \|\tilde{u}(t)\|_1^2 \leqslant M_1\tilde{\rho}_1(\tilde{u}_0, \tilde{u}_1, \tilde{f})\, e^{M_2 t}.$$

Remark 5.2. Theorem 5.1 shows that scheme (2.7) is not stable in the sense of Lax. But if the errors of data are bounded, then for sufficiently large N, the error of numerical solution is still controlled by the errors of data. Indeed, scheme (2.7) is of generalized stability with the index $s \leqslant 0$ (see [8]).

Now we consider the special case with α satisfying (4.5). By taking the inner product with $2\tilde{u}_{\hat{t}}(x, t)$ in the first formula of (5.6), we have from Lemma 7 that

$$\begin{aligned} &(\|\tilde{u}_{\bar{t}}(t)\|^2)_t + (|\tilde{u}(t)|_1^2 + b\,\|\tilde{u}(t)\|^2 + \frac{2}{p}\,\|\tilde{u}(t)\|_{L^p,N}^p)_{\hat{t}} \\ &\quad + 2(\tilde{R}(t), \tilde{u}_{\hat{t}}(t)) = 2(\hat{\tilde{f}}(t), \tilde{u}_{\hat{t}}(t)), \end{aligned} \tag{5.10}$$

where

$$\tilde{R}(t) = P_c(\tilde{G}(t) - G(\tilde{u}(t)))$$

and

$$\begin{aligned} \|R(t)\|^2 &\leqslant d(u)(\|\tilde{u}(t+\tau)\|_1^2 + \|\tilde{u}(t-\tau)\|_1^2 \\ &\quad + \|\tilde{u}(t+\tau)\|_{L^p,N}^p + \|\tilde{u}(t-\tau)\|_{L^p,N}^p). \end{aligned}$$

Similar to the derivation of (5.4), we have from (5.10) that

$$\begin{aligned}(1-ct\tau)\,\|\tilde{u}_t(t)\|^2 &+ \left(\frac{1}{2}-\tau d_s(u)\right)(\|\tilde{u}(t)\|_1^2+\|\tilde{u}(t-\tau)\|_1^2)\\ &+\left(\frac{1}{p}-\tau d_s(u)\right)(\|\tilde{u}(t)\|_{L^p,N}^p+\|\tilde{u}(t-\tau)\|_{L^p,N}^p)\\ &\leqslant \tilde{\rho}_2(\tilde{u}_0,\tilde{u}_1,\tilde{f})+(c+d(u))\tau\\ &\times \sum_{t'\leqslant t-\tau}(\|\tilde{u}_t(t')\|^2+\|\tilde{u}(t')\|^2), \qquad (5.11)\end{aligned}$$

where

$$\tilde{\rho}_2(\tilde{u}_0,\tilde{u}_1,\tilde{f})=(c+d(u_0))(\|\tilde{u}_0\|_1^2+\|\tilde{u}_0\|_1^p+\|\tilde{u}_1\|^2+\|\tilde{u}_1\|^p)+2\tau\sum_{t'\leqslant T}\|\tilde{f}(t')\|_{L^2,N}^2.$$

If N is suitably large, then we can alos check the boundedness of $d(u)$ as before. Thus (5.11) implies that

$$\begin{aligned}\|\tilde{u}_t(t)\|^2+\|\tilde{u}(t)\|_1^2+\|\tilde{u}(t)\|_{L^p}^p &\leqslant M_3\tilde{\rho}_2(\tilde{u}_0,\tilde{u}_1,\tilde{f})\\ &+M_4\tau\sum_{t'\leqslant t-\tau}(\|\tilde{u}_t(t')\|^2+\|\tilde{u}(t')\|_1^2).\end{aligned}$$

Hence we reach at the following theorem.

THEOREM 5.2. *Assume α satisfies* (4.5), $\tau N\leqslant r$, *and N is suitably large. Then for all $t\in R_\tau$ and any $\rho_2(\tilde{u}_0,\tilde{u}_1,\tilde{f})$,*

$$\|\tilde{u}_t(t)\|^2+\|\tilde{u}(t)\|_1^2+\|\tilde{u}(t)\|_{L^p}^p\leqslant M_3\tilde{\rho}_2(t)\,e^{M_4 t}.$$

Remark 5.3. If the conditions of Theorem 5.2 are fulfilled, then scheme (2.7) is of generalized stability with the index $S=-\infty$ (see [8]). It means that there is no restriction on the errors of data. In this case scheme (2.7) is more stable.

VI. CONVERGENCE

In this section, we study the convergence of scheme (2.7). Let $U^{(N)}=P_cU$. Then we have from (1.1) that

$$\begin{aligned}&U_{tt}^{(N)}(x_j,t)-\Delta\hat{U}^{(N)}(x_j,t)+b\hat{U}^{(N)}(x_j,t)\\ &\quad+G(U^{(N)}(x_j,t))=\hat{f}(x_j,t)+\tilde{F}(x_j,t),\\ &\quad x_j\in\Omega_N,\quad t\in S_\tau\\ &U_t^{(N)}(x_j,0)=U_1(x_j)+\tfrac{1}{2}\tau(\Delta U_0(x_j)-bU_0(x_j)\qquad (6.1)\\ &\quad-g(U_0(x_j))+f(x_j,0))+\tilde{g}(x_j),\qquad x_j\in\Omega_N,\\ &U^{(N)}(x_j,0)=U_0(x_j),\qquad x_j\in\Omega_N,\end{aligned}$$

where

$$\begin{aligned}\tilde{F}(x,t)&=P_c\left[U_{tt}^{(N)}(x,t)-\frac{\partial^2\hat{U}^{(N)}}{\partial t^2}(x,t)\right]+P_c[\Delta\hat{U}(x,t)]\\ &\quad-\Delta\hat{U}^N(x,t)+P_c[G(U^{(N)}(x,t))-\hat{g}(U(x,t))]\\ \tilde{g}(x)&=P_c\left[U_t(x,0)-\frac{\partial U}{\partial t}(x,0)-\frac{1}{2}\tau\frac{\partial^2U}{\partial t^2}(x_0)\right].\end{aligned}$$

Let $\tilde{U}=u-U^{(N)}$. Then a subtraction of (6.1) from (2.7) yields that

$$\begin{aligned}&\tilde{U}_{tt}(x_j,t)-\Delta\hat{\tilde{U}}(x_j,t)+b\hat{\tilde{U}}(x_j,t)\\ &\quad+P_c[G(U^{(N)}(x_j,t)+\tilde{U}(x_j,t))-G(U^{(N)}(x_j,t))]\\ &\quad=-\tilde{F}(x_j,t),\qquad x_j\in\Omega_N,\quad t\in R_\tau \qquad (6.2)\\ &\tilde{U}_t(x_j,0)=-\tilde{g}(x_j),\qquad x_j\in\Omega_N,\\ &\tilde{U}(x_j,0)=0,\qquad x_j\in\Omega_N.\end{aligned}$$

Evidently we can obtain an estimation similar to Theorem 5.1 about $\tilde{U}$, where $\|\tilde{u}_t(t)\|$, $\|\tilde{u}(t)\|$ are replaced by $\|\tilde{U}_t(t)\|$ and $\|\tilde{U}(t)\|$, $\tilde{\rho}_1(\tilde{u}_0,\tilde{u}_1,\tilde{f})$, $d(u)$ and $d(\tilde{u})$ by $\tilde{\rho}_1(0,\tilde{g},\tilde{F})$, $d(U)$ and $d(\tilde{U})$, respectively. Thus in order to obtain the convergence rate, we need only to check that $\tilde{\rho}_1(0,\tilde{g},\tilde{F})\leqslant$ const and to estimate its order.

Assume that U is sufficiently smooth. By Lemma 1, Lemma 3, and Lemma 4, we can easily verify that

$$\begin{aligned}2\tau\sum_{t'\leqslant T}\|\tilde{F}(t')\|^2&\leqslant c\tau^4\,\|U\|_{C^4(0,T;L^\infty)}^2\\ &+cN^{-2s}\,\|U\|_{C(0,T;H^{s+2})}^{2\alpha+2}\\ &+c\tau^{2\beta}\,\|U\|_{C^1(0,T;L^\infty)}^{2\alpha+2}\end{aligned}$$

and

$$\|\tilde{g}\|^2\leqslant c\tau^4\,\|U\|_{C^3(0,T;L^\infty)}^2,$$

where $\beta=1$ if $0\leqslant\alpha<1$ and $\beta=2$ if $\alpha\geqslant 1$. Therefore we obtain that

$$\tilde{\rho}_1(0,\tilde{g},\tilde{F})=O(\tau^{2\beta}+N^{-2s}),$$

provided that $U\in C(0,T;H_p^{s+2})\cap C^4(0,T;L^\infty)$.

Finally by the triangle inequality and

$$\|U_t^N(t)-U_t(t)\|^2\leqslant cN^{-2s}\left\|\frac{\partial U}{\partial t}\right\|_{C(0,T;H^s)},\qquad \text{if}\quad U\in C^1(0,T;H_p^s),$$

we obtain the following theorem.

THEOREM 6.1. *Assume* $\tau N \leqslant r < \infty$ *and the exact solution* $U \in C(0, T; H_p^{s+2}) \cap C^1(0, T; H_p^s) \cap C^4(0, T; L^\infty)$. *Then for N large enough, we have*

$$\|U_t(t) - u_t(t)\|^2 + \|U(t) - u(t)\|_1^2 = O(\tau^{2\beta} + N^{-2s}).$$

Remark 6.1. The above estimation for the convergence rate is not optimal. This is caused by our comparison between $u(t)$ and $P_c U(t)$ in the proof, which generates the term $P_c(\Delta U(t)) - \Delta(P_c U(t))$, lowering the convergence rate. However, if $g(U) \in C(0, T; H_p^s)$, as in the case where α is an integer, then we can compare $u(t)$ with $P_N U(t)$, the L^2 orthogonal projection of $U(t)$ onto S_N, instead. In this way, the requirement $U \in C(0, T; H_p^{s+2})$ can be relaxed, and the optimal convergence rate will result.

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